AN URBAN ECONOMIC MODEL OVER A CONTINUOUS PLANE WITH SPATIAL CHARACTERISTIC VECTOR FIELD
- ASYMMETRIC LAND USE PATTERN AND INTERNALIZING TRANSPORTATION NETWORKS -

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Abstract
Among others Beckmann (1952) firstly introduced the concept of a two dimensional continuous space into economics. This great step had unfortunately not shown further expansion in economics. Through several papers related to Beckmann's initiation, Beckmann and Puu (1985) at last reached a systematic treatment of the continuous spatial economics. Although their achievement is fascinated by employing a partial differential equations approach, Beckmann's original philosophy, that is, the gradient law has still been inherited. Beckmann and Puu's book (1985) aims to study formation of urban configuration in a two dimensional continuous space, focusing on flows of commodities. However, consideration of households and firms location is not necessarily sufficient, resulting in reconsideration from a new urban economics point of view. Differing from Beckmann and Puu's studies, Miyata (2010) introduces bid rent functions (Fujita (1989)), which are familiar in the new urban economics, for land of households and firms, and then he studies how the results of Beckmann and Puu are rigorously modified by using the theory of partial differential equations. However Miyata (2010) deals with a symmetric equilibrium which seems to be a little unrealistic. This article extends the author's previous study introducing spatial characteristic vector field in the model which stands for heterogeneity in geographical conditions in a city, and try to show asymmetry in land use pattern and endogenous formation of transportation networks in a two dimensional continuous space.

Keywords continuous plane city · Beckmann · Puu · asymmetric land use · heterogeneity ·

JEL, **classification** R10, R40, R30

1. INTRODUCTION

Among others Beckmann (1952) firstly introduced the concept of a two dimensional continuous space into economics. This great step had unfortunately not shown further expansion in economics. Through several papers related to Beckmann's initiation, Beckmann and Puu (1985) at last reached a systematic treatment of the continuous spatial economics. Although their achievement is fascinated by employing a partial differential equations approach, Beckmann's original philosophy, that is, the gradient law has still been inherited. Following their achievement, Puu (2003) alone developed their theory by using many computer simulations to visually show the significance of their theory.

Beckmann and Puu's book (1985) aims to study formation of urban configuration in a two dimensional continuous space, focusing on flows of commodities. However, consideration of households and firms location is not necessarily sufficient, resulting in reconsideration from a new urban economics point of view. As another exceptional urban economic study of a plane city, Lucus and Rossi-Hansberg (2002) is pointed out. They were inspired by Fujita and Ogawa (1982), and indicate endogenous land use pattern over a plane city. However they neglect commodity market to simplify the analysis. Well discussion about formation of urban configuration is summarized in Anas, Arnott, and Small (1998).

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Differing from Beckmann and Puu's studies, Miyata (2010) introduces bid rent functions (Fujita (1989)), which are familiar in the new urban economics, for land of households and firms, and then it studies how the results of Beckmann and Puu are rigorously modified by using the theory of partial differential equations (Courant and Hilbert (1953, 1962).

However Miyata (2010) deals with a symmetric equilibrium which seems to be a little unrealistic. This article extends the author's previous study introducing spatial characteristic vector field in the model which stands for heterogeneity in geographical conditions in a city, and try to show asymmetry in land use pattern and endogenous formation of transport networks in a two dimensional continuous space.

2. ASSUMPTIONS OF THE MODEL

(1)There is a city in a two dimensional continuous space (a plane). The land characteristics are heterogeneous. The characteristics of each point are denoted by $(lc_1(x_1, x_2),\cdots,lc_n(x_1, x_2))$ named spatial characteristic vector field (SCVF). For example, one can consider uneven land and/or a gradient. SCVF gives impact only on transport cost.
(2)Transports of commodities and labor are considered in this study. The transportation cost is expressed as von Thünen type. The transport cost of unit volume of commodity at point (x_1, x_2) is specified as a scalar function over SCVF denoting it by $\varepsilon(x_1, x_2)$. Regarding workers commuting, transport cost per head and per distance is denoted by $\zeta(x_1,x_2)$.
(3)There are N households and M firms in the city. N and M denote the numbers of households and firms, respectively, and they are sufficiently large. Households and firms are homogeneous, respectively.
(4)Land in the city is owned by absentee landowners, who reside outside the city. There is a unique local government in the city, and it rents all land in the city from the absentee landowners. The local government rents the land to households and firms at market rent, and then redistributed it equally to households.
(5)The service of capital stock is assumed to be the numerare, since the capital stock is assumed to move in free of cost in the city leading to unique equilibrium capital return rate being irrespective firms' locations.
(6)As mentioned later, the parameters in the locational potential function for each firm is sufficiently large. In this case a simple von Thünen ring becomes an equilibrium urban configuration.

3. FIRMS BEHAVIOR

The production function of a firm at location $\boldsymbol{x} = (x_1, x_2)$ is specified as a Cobb-Douglas type of homogeneous degree of unity, and agglomeration economy is taken into account. The agglomeration economy is represented by locational potential function, $\Omega(\boldsymbol{x})$, which was introduced by Fujita and Ogawa (1982). The locational potential function is defined as follows:

$$\Omega(\boldsymbol{x}) \equiv \int\int_A \mu b(\boldsymbol{y})\exp(-\omega \| \boldsymbol{x}-\boldsymbol{y} \|)dy_1 dy_2 \quad (1)$$

where
A : city area
$b(\boldsymbol{y})$: density of firms at location $\boldsymbol{y}$
μ : monetary conversion parameter in locational potential
ω : parameter expressing the effects of distance between different points
$\| \boldsymbol{x}-\boldsymbol{y} \|$: distance between location $\boldsymbol{x}$ and $\boldsymbol{y}$

Then the production by a firm at location $\boldsymbol{x}$ may be written as follows:

$$q(\boldsymbol{x}) = \Omega(\boldsymbol{x}) q_A k_d(\boldsymbol{x})^{\alpha_k} l_d(\boldsymbol{x})^{\alpha_l} m_B(\boldsymbol{x})^{\alpha_m} \quad (2)$$

where
$q(\boldsymbol{x})$: output of a firm at location $\boldsymbol{x}$,
$k_d(\boldsymbol{x})$: input of capital stock at location $\boldsymbol{x}$
$l_d(\boldsymbol{x})$: input of labor at location $\boldsymbol{x}$
$m_B(\boldsymbol{x})$: input of land at location $\boldsymbol{x}$

q_A : efficient parameter in the production function
α_k, α_l, α_m : elasticity parameters in the production function ($\alpha_k + \alpha_l + \alpha_m = 1$)

We assume that each firm is a price taker for commodities and production factors. The firms' locational equilibrium condition is that the profit in each firm is equalized at every point. Due to the linear homogeneity of degree one in each firm's technology, the equilibrium profit in each firm becomes zero. Then the bid rent function in each firm is defined as follows:

$$g_B(\boldsymbol{x}) \equiv max\left[\frac{p(\boldsymbol{x})q(\boldsymbol{x}) - r(\boldsymbol{x})k_d(\boldsymbol{x}) - w(\boldsymbol{x})l_d(\boldsymbol{x})}{m_B(\boldsymbol{x})}\right] \tag{3}$$

with respect to $k_d(\boldsymbol{x}), l_d(\boldsymbol{x})$ *and* $m_B(\boldsymbol{x})$
subject to

$$q(\boldsymbol{x}) = \Omega(\boldsymbol{x})q_A k_d(\boldsymbol{x})^{\alpha_k} l_d(\boldsymbol{x})^{\alpha_l} m_B(\boldsymbol{x})^{\alpha_m}$$

$$\pi_B(\boldsymbol{x}) = 0$$

where
$p(\boldsymbol{x})$: price of commodity at location $\boldsymbol{x}$
$r(\boldsymbol{x})$: capital return rate at location $\boldsymbol{x}$,
$w(\boldsymbol{x})$: wage rate at location $\boldsymbol{x}$
$g_B(\boldsymbol{x})$: bid rent by a firm at location $\boldsymbol{x}$
$\pi_B(\boldsymbol{x})$: profit in a firm at location $\boldsymbol{x}$

Solving this optimization problem, we obtain the conditional demands for labor and capital associated with firm's output $q(\boldsymbol{x})$, the bid max lot size and the bid rent function.

$$k_d(\boldsymbol{x}) = \frac{\alpha_k p(\boldsymbol{x})}{r(\boldsymbol{x})} q(\boldsymbol{x}) \tag{4}$$

$$l_d(\boldsymbol{x}) = \frac{\alpha_l p(\boldsymbol{x})}{w(\boldsymbol{x})} q(\boldsymbol{x}) \tag{5}$$

$$m_B(\boldsymbol{x}) = (\Omega(\boldsymbol{x})q_A)^{-\frac{1}{\alpha_m}} p(\boldsymbol{x})^{\frac{\alpha_m - 1}{\alpha_m}} \left[\frac{r(\boldsymbol{x})}{\alpha_k}\right]^{\frac{\alpha_k}{\alpha_m}} \left[\frac{w(\boldsymbol{x})}{\alpha_l}\right]^{\frac{\alpha_l}{\alpha_m}} q(\boldsymbol{x}) \tag{6}$$

$$g_B(\boldsymbol{x}) = \alpha_m (p(\boldsymbol{x})\Omega(\boldsymbol{x})q_A)^{\frac{1}{\alpha_m}} \left[\frac{\alpha_k}{r(\boldsymbol{x})}\right]^{\frac{\alpha_k}{\alpha_m}} \left[\frac{\alpha_l}{w(\boldsymbol{x})}\right]^{\frac{\alpha_l}{\alpha_m}} \tag{7}$$

4. HOUSEHOLDS

The household utility function at location $\boldsymbol{x}$ is assumed to be expressed as follows:

$$u(c(\boldsymbol{x}), m_H(\boldsymbol{x})) \equiv c(\boldsymbol{x})^{\beta_c} m_H(\boldsymbol{x})^{\beta_m} \quad (\beta_c + \beta_m = 1) \tag{8}$$

where
$u(c(\boldsymbol{x}), m_H(\boldsymbol{x}))$: household utility function at location $\boldsymbol{x}$
$c(\boldsymbol{x})$: household consumption at location $\boldsymbol{x}$
$m_H(\boldsymbol{x})$: household land input at location $\boldsymbol{x}$
β_c and β_m : elasticity parameters in the utility function

Each household endows available working time l_s, and capital stock k_s, and is assumed to inelastically supply them to firms obtaining income of $w(\boldsymbol{x})\, l_s + r(\boldsymbol{x})\, k_s$ plus redistributed income from the local government π_H. In household locational equilibrium, the household utility level takes the same value u^*, being irrespective of household residential places. Therefore the household bid rent function for land is specified as follows:

$$g_H(\boldsymbol{x}) \equiv max \frac{w(\boldsymbol{x})l_s + r(\boldsymbol{x})k_s + \pi_H - p(\boldsymbol{x})c(\boldsymbol{x})}{m_H(\boldsymbol{x})} \tag{9}$$

with respect to $c(\boldsymbol{x})$ *and* $m_H(\boldsymbol{x})$

subject to $u(\boldsymbol{x}) = u^*$
where $g_H(\boldsymbol{x})$ is a household bid rent function at location $\boldsymbol{x}$, and π_H is defined as follows:

$$\pi_H \equiv \frac{1}{N}\int\int_A [g_B(\boldsymbol{x})b(\boldsymbol{x})m_B(\boldsymbol{x}) + g_H(\boldsymbol{x})h(\boldsymbol{x})m_H(\boldsymbol{x})]dx_1dx_2 \quad (10)$$

where
$h(\boldsymbol{x})$: density of households at location $\boldsymbol{x}$

Solving the optimization problem (9), one obtains per capita consumption, the bid max lot size and the bid rent function.

$$c(\boldsymbol{x}) = \frac{\beta_c}{p(\boldsymbol{x})}(w(\boldsymbol{x})l_s + r(\boldsymbol{x})k_s + \pi_H) \quad (11)$$

$$m_H(\boldsymbol{x}) = \left[\frac{p(\boldsymbol{x})}{\beta_c(w(\boldsymbol{x})l_s + r(\boldsymbol{x})k_s + \pi_H)}\right]^{\frac{\beta_c}{\beta_m}} u^{*\frac{1}{\beta_m}} \quad (12)$$

$$g_H(\boldsymbol{x}) = \beta_m\left[\frac{\beta_c}{p(\boldsymbol{x})}\right]^{\frac{\beta_c}{\beta_m}}\left[\frac{w(\boldsymbol{x})l_s + r(\boldsymbol{x})k_s + \pi_H}{u^*}\right]^{\frac{1}{\beta_m}} \quad (13)$$

5. LAND MARKET EQUILIBRIUM CONDITION

We consider a monocentric land use patter, since it has been demonstrated by Miyata (2010) that a monocentric land use patter becomes an equilibrium one if parameters in the locational function are appropriately chosen. The market rent function is denoted by;

$$g(\boldsymbol{x}) \equiv max\{g_B(\boldsymbol{x}), g_H(\boldsymbol{x}), g_A\} \quad (14)$$

where g_A stands for agricultural land rent exogenously given.

If the business district is located surrounding the city center, and the residential area exists surrounding the business district, the equilibrium land rent is denoted as follows:

$g(\boldsymbol{x}) = g_B(\boldsymbol{x}) \geq g_H(\boldsymbol{x})$: $\boldsymbol{x} \in$ business district (15)
$g(\boldsymbol{x}) = g_B(\boldsymbol{x}) = g_H(\boldsymbol{x})$: $\boldsymbol{x} \in$ boundary between the business district and the residential area (16)
$g(\boldsymbol{x}) = g_H(\boldsymbol{x}) \geq g_B(\boldsymbol{x})$: $\boldsymbol{x} \in$ the residential area (17)
$g(\boldsymbol{x}) = g_H(\boldsymbol{x}) = g_A(\boldsymbol{x})$: $\boldsymbol{x} \in$ on the city boundary (18)

The constraints on the numbers of firms and the households are as follows:

$$M = \int\int_{A_B} \frac{1}{m_B(\boldsymbol{x})}dx_1dx_2 \quad \text{where } A_B \text{ is business district.} \quad (19)$$

$$N = \int\int_{A_H} \frac{1}{m_H(\boldsymbol{x})}dx_1dx_2 \quad \text{where } A_H \text{ is residential area.} \quad (20)$$

6. LOCAL BALANCE EQUATIONS FOR COMMODITY, LABOR, AND CAPITAL STOCK

Let us assume the transport technology for commodities as von Thünen technology with a variable ratio $\varepsilon(\boldsymbol{x}) \equiv \varepsilon(x_1, x_2)$ which depends on location. That is, the transport cost is incurred in the shipped commodities being expressed as $\varepsilon(\boldsymbol{x})\cdot q$ in shipping q unit of commodities in the unit distance. Let $\varphi(\boldsymbol{x})$ be a two dimensional vector of commodities transported to location $\boldsymbol{x}$ in unit time and in unit area size. Then the following local balance equation for commodities is realized (Beckmann and Puu (1985)).

$$div\,\varphi(\boldsymbol{x}) = q(\boldsymbol{x})b(\boldsymbol{x}) - c(\boldsymbol{x})h(\boldsymbol{x}) - \varepsilon(\boldsymbol{x})\|\varphi(\boldsymbol{x})\| \quad (21)$$

where
$b(\boldsymbol{x})$: firms' density at location $\boldsymbol{x}$ ($= 1 / m_B(\boldsymbol{x})$)
$h(\boldsymbol{x})$: households' density at location $\boldsymbol{x}$ ($= 1 / m_H(\boldsymbol{x})$)

$\|\cdot\|$: norm of a two dimensional vector

Similarly, let $\zeta(\boldsymbol{x}) \equiv \zeta(x_1, x_2)$ denote the transport cost in conveying unit labor in unit distance. This cost is incurred in labor itself as well. Let $\psi(\boldsymbol{x})$ express a two dimensional vector of labor in unit time and in unit area size shipped to location $\boldsymbol{x}$. Then the local balance equation for labor held at location $\boldsymbol{x}$ is expressed as follows (Beckmann and Puu (1985)):

$$div\,\psi(\boldsymbol{x}) = l_s h(\boldsymbol{x}) - l_d(\boldsymbol{x})b(\boldsymbol{x}) - \xi(x)\,\|\,\psi(\boldsymbol{x})\,\| \tag{22}$$

As for capital stock, we assume that capital stock can move to arbitrary places in free of cost. This assumption reflects the fact that if once the capital stock is installed in firms, spatial movement of capital stock after the installation can be considered to be less than commodities and labor. Thus the equilibrium in capital stock holds at the location of each firm. This is described as follows:

$$k_d(\boldsymbol{x})b(\boldsymbol{x}) = ks(\boldsymbol{x}) \tag{23}$$

where $\int\int_A ks(\boldsymbol{x})dx_1dx_2 = \int\int_A k_s h(\boldsymbol{x})dx_1dx_2$ and A stands for the city area.

Due to the free of cost assumption in capital stock location, the following formula holds.

$$\int\int_A ks(\boldsymbol{x})dx_1dx_2 = k_s N \tag{24}$$

where N : the number of households residing in the city

7. THE LOCAL GOVERNMENT AND ABSENTEE LANDOWNERS

The city area is assumed to be occupied by absentee landowners. Absentee landowners reside outside of the city. The local government rents all land in the city from absentee landowners. It rents the land to firms and households in the city at market rents. The revenue of the local government from the land rent is equally redistributed to households. The redistributed household income, π_H, is indicated as formula (10) mentioned earlier.

8. GLOBAL EQUILIBRIUM CONDITION IN COMMODITY AND LABOR MARKETS

Integrating the commodity local balance equation (21), and applying Gauss divergence theorem, one can obtain the global equilibrium condition on commodities as presented in equation (26), since all commodities produced in the city are consumed by households in the city, and we assume that there is no import and export of commodities. So the right hand side in equation (26) becomes zero.

$$\int\int_A div\,\varphi(\boldsymbol{x})dx_1dx_2 = \int\int_A [q(\boldsymbol{x})b(\boldsymbol{x}) - c(\boldsymbol{x})h(\boldsymbol{x}) - \varepsilon(\boldsymbol{x})\,\|\,\varphi(\boldsymbol{x})\,\|]dx_1dx_2 = \int_{\partial A} c_n(\boldsymbol{x}(s))ds = 0 \tag{25}$$

$$\therefore \int\int_A [q(\boldsymbol{x})b(\boldsymbol{x}) - c(\boldsymbol{x})h(\boldsymbol{x}) - \varepsilon(\boldsymbol{x})\,\|\,\varphi(\boldsymbol{x})\,\|]dx_1dx_2 = 0 \tag{26}$$

where

∂A: city boundary

$c_n(\boldsymbol{x}(s))$: net export whose direction is normal to the city boundary

Next integrating the labor local balance equation (22) with Gauss divergence theorem, one can obtain the global equilibrium equation for labor. Since we assume that there is no in- and out-migration at the city boundary, the line integral of labor migration along the city boundary becomes zero as well.

$$\int\int_A div\,\psi(\boldsymbol{x})dx_1dx_2 = \int\int_A [l_s h(\boldsymbol{x}) - l_d(\boldsymbol{x})b(\boldsymbol{x}) - \xi(\boldsymbol{x})\,\|\,\psi(\boldsymbol{x})\,\|]dx_1dx_2 = \int_{\partial A} l_n(\boldsymbol{x}(s))ds = 0 \tag{27}$$

$$\therefore \int\int_A [l_s h(\boldsymbol{x}) - l_d(\boldsymbol{x})b(\boldsymbol{x}) - \xi(\boldsymbol{x})\,\|\,\psi(\boldsymbol{x})\,\|]dx_1dx_2 = 0 \tag{28}$$

9. COMMODITY TRANSPORT AGENT (*C.T.A.*)

Transporting commodities is assumed to be done by the commodity transport agent (*C.T.A.*) (Beckmann and Puu (1985)). The *C.T.A.* buys $p(\boldsymbol{x})q(\boldsymbol{x})b(\boldsymbol{x})$ of commodities at point $\boldsymbol{x}$, and sells $p(\boldsymbol{x})c(\boldsymbol{x})h(\boldsymbol{x})$ of commodities to households. Thus the profit of *C.T.A.* at $\boldsymbol{x}$ is expressed as follows:

$$\pi_{Tq}(\boldsymbol{x}) = p(\boldsymbol{x})c(\boldsymbol{x})h(\boldsymbol{x}) - p(\boldsymbol{x})q(\boldsymbol{x})b(\boldsymbol{x}) = -p(\boldsymbol{x})div\,\varphi(\boldsymbol{x}) - p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\,\|\,\varphi(\boldsymbol{x})\,\| \tag{29}$$

The *C.T.A.* aims to find the optimal route which maximizes the profit earned over the entire city area. The profit of the *C.T.A.* over the entire city area is written as follows:

$$\iint_A \pi_{Tq}(\boldsymbol{x})dx_1dx_2 = \iint_A [p(\boldsymbol{x})c(\boldsymbol{x})h(\boldsymbol{x}) - p(\boldsymbol{x})q(\boldsymbol{x})b(\boldsymbol{x})]dx_1dx_2$$

$$= -\iint_A [p(\boldsymbol{x})div\,\varphi(\boldsymbol{x}) + p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\|\varphi(\boldsymbol{x})\|]dx_1dx_2 \quad (30)$$

The necessary and sufficient condition for the profit maximization in the *C.T.A.* is derived from the calculus variation (Gelfand and Fomin (1963)). The Euler-Lagrange equation in the calculus variation is as follows:

$$\frac{d}{dx_1}\frac{\partial \pi_{Tq}}{\partial(\partial\varphi_i/\partial x_1)} + \frac{d}{dx_2}\frac{\partial \pi_{Tq}}{\partial(\partial\varphi_i/\partial x_2)} - \frac{\partial \pi_{Tq}}{\partial \varphi_i} = 0 \quad (31)$$

where $\boldsymbol{x} = (x_1, x_2)$ and $\varphi(\boldsymbol{x}) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$.

Transforming the profit in the *C.T.A.* in the x_1-x_2 coordinate, we have;

$$\pi_{Tq}(\boldsymbol{x}) = -p(\boldsymbol{x})(div\,\varphi(\boldsymbol{x}) - p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\|\varphi(\boldsymbol{x})\|) = -p(x_1,x_2)\left\{\frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2} + \varepsilon(\boldsymbol{x})(\varphi_1^2 + \varphi_2^2)^{1/2}\right\} \quad (32)$$

Therefore the Euler-Lagrange equation is concretely expressed as follows:

$$\frac{\partial \pi_{Tq}}{\partial(\partial\varphi_i/\partial x_i)} = -p(\boldsymbol{x}) \quad (33)$$

$$\frac{d}{dx_i}\frac{\partial \pi_{Tq}}{\partial(\partial\varphi_i/\partial x_i)} = -\frac{\partial p(\boldsymbol{x})}{\partial x_i} \quad (34)$$

$$\frac{\partial \pi_{Tq}}{\partial \varphi_i} = -p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\frac{\varphi_i(\boldsymbol{x})}{\|\varphi(\boldsymbol{x})\|} \quad (35)$$

Thus the optimal condition is finally reduced to equation (36).

$$\therefore p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\frac{\varphi(\boldsymbol{x})}{\|\varphi(\boldsymbol{x})\|} = grad\ p(\boldsymbol{x}) \quad (36)$$

In equation (36), $\varphi(\boldsymbol{x}) / \|\varphi(\boldsymbol{x})\|$ stands for the direction along which commodities are transported, and that direction coincides the gradient of commodity price. This condition is the optimal one for commodity transport. In other words, the direction along which commodities are transported is the direction where commodity price gets highest. This is famous a finding as the so-called gradient law by Beckmann (1952).

$p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\varphi(\boldsymbol{x}) / \|\varphi(\boldsymbol{x})\|$ depicts the cost of shipping a unit commodity in a unit distance. Let us calculate the transport cost in carrying a unit commodity from the point $\boldsymbol{x}_A$ to $\boldsymbol{x}_B$ on the optimal route. We denote the route by $D(s) = (x_1(s), x_2(s))$ $(0 \le s \le 1$, $\boldsymbol{x}_A = (x_1(0), x_2(0))$, $\boldsymbol{x}_B = (x_1(1), x_2(1)))$. Thus the transport cost is expressed as follows:

$$\int_0^1 p(\boldsymbol{x}(s))a\frac{\varphi(\boldsymbol{x}(s))}{\|\varphi(\boldsymbol{x}(s))\|}\frac{d\boldsymbol{x}(s)}{ds}ds = \int_0^1 grad\ p(\boldsymbol{x}(s))\frac{d\boldsymbol{x}(s)}{ds}ds = p(\boldsymbol{x}_B) - p(\boldsymbol{x}_A) \quad (37)$$

This equation implies that the transport cost in carrying the unit commodity along the optimal route is the difference between commodity prices at different points. This also asserts that the transport cost is incurred in commodity price. Then let us calculate the profit in the *C.T.A.* earned in the entire city area. Multiplying the both hand sides in equation (36) by $\varphi(\boldsymbol{x})$ in the sense of scalar product, we have;

$$p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\|\varphi(\boldsymbol{x})\| = \varphi(\boldsymbol{x})grad\ p(\boldsymbol{x}) \quad (38)$$

$$\therefore \iint_A \pi_{Tq}(\boldsymbol{x})dx_1dx_2 = -\iint_A [p(\boldsymbol{x})div\,\varphi(\boldsymbol{x}) + p(\boldsymbol{x})\varepsilon(\boldsymbol{x})\|\varphi(\boldsymbol{x})\|]dx_1dx_2$$

$$= -\iint_A [p(\boldsymbol{x})div\,\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{x})grad\ p(\boldsymbol{x})]dx_1dx_2 \quad (39)$$

By the way, $div\ p(\boldsymbol{x})\varphi(\boldsymbol{x}) = p(\boldsymbol{x})div\,\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{x})grad\ p(\boldsymbol{x})$ holds. Thus equation (39) can further be transformed as;

$$\iint_A \pi_{Tq}(\boldsymbol{x})dx_1dx_2 = -\iint_A div\ p(\boldsymbol{x})\varphi(\boldsymbol{x})dx_1dx_2 = -\int_{\partial A} p_n(\boldsymbol{x}(s))c_n(\boldsymbol{x}(s))ds = 0 \quad (40)$$

Equation (40) asserts that the total profit in the *C.T.A.* in the city area becomes zero.

10. LABOR TRANSPORT AGENT (*L.T.A.*)

In turn, we formulate the behavior of the labor transport agent (*L.T.A.*) (Beckmann and Puu (1985)). The *L.T.A.* receives wages of $w(\boldsymbol{x})l_d(\boldsymbol{x})b(\boldsymbol{x})$ from firms at location $\boldsymbol{x}$, and pays wages of $w(\boldsymbol{x})l_s h(\boldsymbol{x})$ to households. As mentioned earlier, it is assumed that there is no migration from and to the outside of the city. Therefore the profit in the *L.T.A.* at $\boldsymbol{x}$ is denoted as follows:

$$\pi_{Tl}(\boldsymbol{x}) = w(\boldsymbol{x})l_d(\boldsymbol{x})b(\boldsymbol{x}) - w(\boldsymbol{x})l_s h(\boldsymbol{x}) = -w(\boldsymbol{x})div\,\psi(\boldsymbol{x}) - w(\boldsymbol{x})\varsigma(\boldsymbol{x})\|\psi(\boldsymbol{x})\| \tag{41}$$

The *L.T.A.* determines labor transport routes so as to maximize the profit which will be gained over the entire city area. This problem is specified as follows:

$$\begin{aligned} max \iint_A \pi_{Tl}(\boldsymbol{x})dx_1dx_2 &= max \iint_A [w(\boldsymbol{x})l_d(\boldsymbol{x})b(\boldsymbol{x}) - w(\boldsymbol{x})l_s h(\boldsymbol{x})]dx_1dx_2 \\ &= max \; -\iint_A [w(\boldsymbol{x})div\,\psi(\boldsymbol{x}) + w(\boldsymbol{x})\varsigma(\boldsymbol{x})\|\psi(\boldsymbol{x})\|]dx_1dx_2 \end{aligned} \tag{42}$$

Similar to the case of commodity transport, the following profit maximization condition is obtained by applying the calculus of variation again.

$$w(\boldsymbol{x})\varsigma(\boldsymbol{x})\frac{\psi(\boldsymbol{x})}{\|\psi(\boldsymbol{x})\|} = grad\; w(\boldsymbol{x}) \tag{43}$$

This equation shows that labor is transported along the direction where the wage rate gets highest as well. Taking two points, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, on the optimal route, the transport cost in moving the unit labor from $\boldsymbol{x}_A$ to $\boldsymbol{x}_B$ along the optimal route is calculated as follows:

$$\int_0^1 w(\boldsymbol{x}(s))\varsigma(\boldsymbol{x}(s))\frac{\psi(\boldsymbol{x}(s))}{\|\psi(\boldsymbol{x}(s))\|}\frac{d\boldsymbol{x}(s)}{ds}ds = \int_0^1 grad\; w(\boldsymbol{x}(s))\frac{d\boldsymbol{x}(s)}{ds}ds = w(\boldsymbol{x}_B) - w(\boldsymbol{x}_A) \tag{44}$$

One can know from equation (44) that the wage rate includes transport cost. The total profit of *L.T.A.* in the city is expressed as follows by using Gauss divergence theorem.

$$\iint_A \pi_{Tl}(\boldsymbol{x})dx_1dx_2 = -\iint_A [w(\boldsymbol{x})div\,\psi(\boldsymbol{x}) + w(\boldsymbol{x})\varsigma(\boldsymbol{x})\|\psi(\boldsymbol{x})\|]dx_1dx_2 = -\int_{\partial A} w_n(\boldsymbol{x}(s))\psi_n(\boldsymbol{x}(s))ds \tag{45}$$

where

$w_n(\boldsymbol{x}(s))$: wage rate prevailing on the city boundary

$\psi_n(\boldsymbol{x}(s))$: volume of labor flow from or to outside of the city

Since we assume that there is no labor flow from and to the outside of the city in this study, $\int_{\partial A} w_n(\boldsymbol{x}(s))\psi_n(\boldsymbol{x}(s))ds = 0$ holds. That is, the zero profit condition holds in the *L.T.A.* as well.

11. WALRAS LAW

Now let us write the budget conditions on all economic agents.

$$\textbf{firms}\;:\; \iint_A [p(\boldsymbol{x})q(\boldsymbol{x}) - r(\boldsymbol{x})k_d(\boldsymbol{x}) - w(\boldsymbol{x})l_d(\boldsymbol{x}) - g_B(\boldsymbol{x})m_B(\boldsymbol{x})]b(\boldsymbol{x})dx_1dx_2 = 0 \tag{46}$$

$$\textbf{households}\;:\; \iint_A [w(\boldsymbol{x})l_s + r(\boldsymbol{x})k_s + \pi_H - p(\boldsymbol{x})c(\boldsymbol{x}) - g_H(\boldsymbol{x})m_H(\boldsymbol{x})]h(\boldsymbol{x})dx_1dx_2 = 0 \tag{47}$$

$$\textbf{local government}\;:\; \iint_A [g_B(\boldsymbol{x})m_B(\boldsymbol{x})b(\boldsymbol{x}) + g_H(\boldsymbol{x})m_H(\boldsymbol{x})h(\boldsymbol{x})]dx_1dx_2 - N\pi_H = 0 \tag{48}$$

Summing up equations (46) to (48), we obtain the following equation.

$$\begin{aligned} &\iint_A p(\boldsymbol{x})[q(\boldsymbol{x})b(\boldsymbol{x}) - c(\boldsymbol{x})h(\boldsymbol{x})]dx_1dx_2 \\ &+\iint_A w(\boldsymbol{x})[l_s h(\boldsymbol{x}) - l_d(\boldsymbol{x})b(\boldsymbol{x})]dx_1dx_2 \\ &+\iint_A r(\boldsymbol{x})[k_s h(\boldsymbol{x}) - k_d(\boldsymbol{x})b(\boldsymbol{x})]dx_1dx_2 = 0 \end{aligned} \tag{49}$$

Equation (49) is interpreted as a kind of Walras law, however, the true Walras law must be more restricted. This point will be referred again in a later part. Any way, taking into account the free mobility of capital stock, that is, the capital rental price is independent of location, we set the capital stock service as numeraire in the model, i.e. $r(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in A$.

12. COMMODITY PRICE

The commodity price equation (36) can be transformed into the following non-linear first order partial differential equation.

$$\left(\frac{\partial ln\, p(\boldsymbol{x})}{\partial x_1}\right)^2 + \left(\frac{\partial ln\, p(\boldsymbol{x})}{\partial x_2}\right)^2 = \varepsilon(\boldsymbol{x})^2 \quad (50)$$

Equation (50) can mathematically be solved, and its solution is formula (51) when the initial manifold is degenerated to a single point.

$$p(x_1, x_2) = p_0 \exp \varepsilon(x_1, x_2)\sqrt{x_1^2 + x_2^2} \quad (51)$$

The initial price p_0 is obtained by the price on the city boundary determined by general equilibrium in the entire city. The formula (51) is called integral conoid showing a cone shaped surface in the three dimensional space as presented Miyata (2010), if $\varepsilon(x_1, x_2)$ is a constant (see Figure 1). Regarding variable $\varepsilon(x_1, x_2)$, its graphical illustration will be shown later.

13. WAGE RATE

Regarding the wage rate, the same discussion is possible. The solution surface for the wage rate is expressed by;

$$w(x_1, x_2) = w_0 \exp(-\varsigma(x_1, x_2)\sqrt{x_1^2 + x_2^2}) \quad (52)$$

When we assume that the business district encircles the origin, and the residential area is located outside the business area, that is, a simple von Thünen ring, the wage rate prevailing at the city center w_0 is given by the wage rate on the city boundary which is determined to satisfy the equilibrium conditions mentioned later. Conversely to the commodity price surface, the wage profile shows a decrease with exponential order from the city center to the city boundary if $\zeta(x_1, x_2)$ is a constant (see Figure 2).

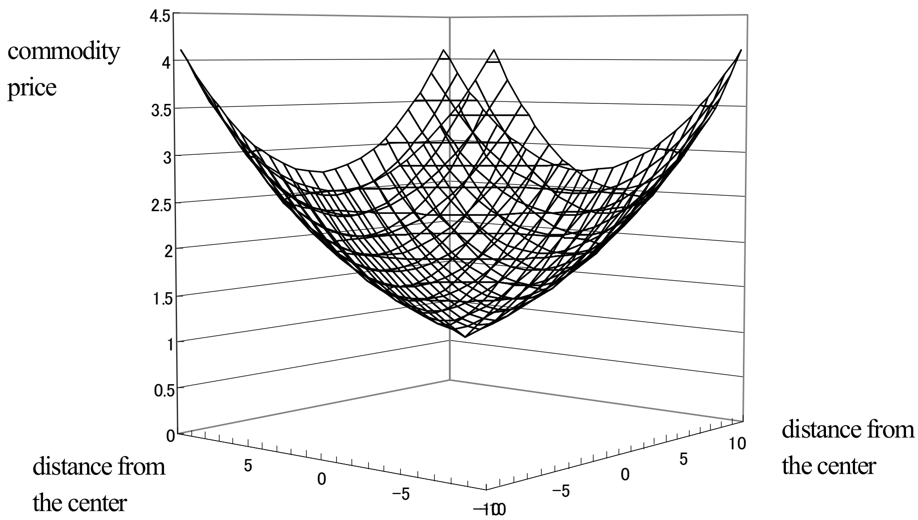

Figure 1 Shape of the Commodity Price Surface

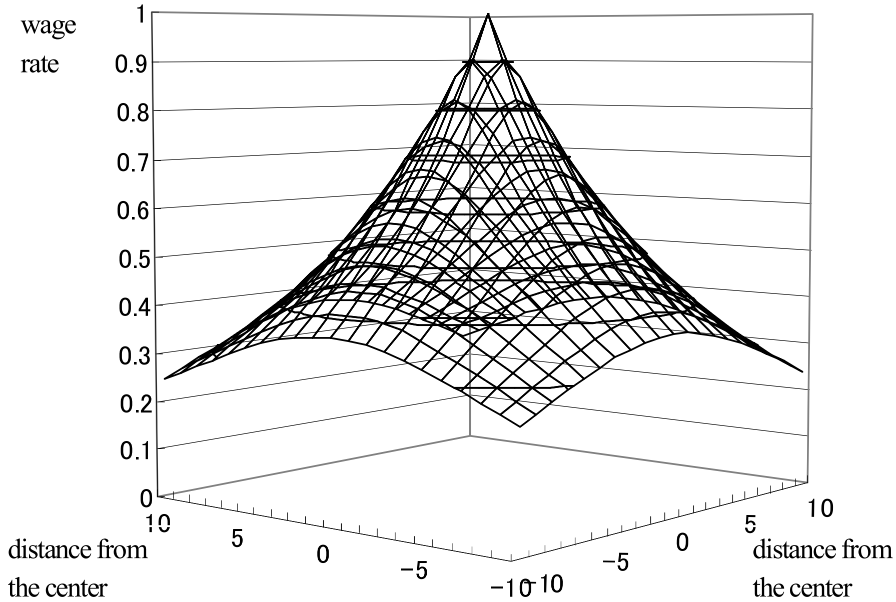

Figure 2 Shape of the Wage Rate Surface

14. THE OPTIMAL TRANSPORT OF COMMODITIES

In this section, we try to solve the optimal volume of shipment of commodities. To this end, we start with transforming equation (43) yielding;

$$\frac{\varphi(\boldsymbol{x})}{\|\varphi(\boldsymbol{x})\|}=\frac{1}{\varepsilon(\boldsymbol{x})}\frac{grad\ p(\boldsymbol{x})}{p(\boldsymbol{x})}=\frac{1}{\varepsilon(\boldsymbol{x})}grad\ ln\ p(\boldsymbol{x}) \tag{53}$$

Therefore $\varepsilon(\boldsymbol{x})^2=\|grad\ ln\ p(\boldsymbol{x})\|^2$ holds, leading to $\varepsilon(\boldsymbol{x})=\|grad\ ln\ p(\boldsymbol{x})\|$.

$$\therefore\frac{\varphi(\boldsymbol{x})}{\|\varphi(\boldsymbol{x})\|}=\frac{grad\ ln\ p(\boldsymbol{x})}{\|grad\ ln\ p(\boldsymbol{x})\|} \tag{54}$$

$$\therefore\varphi(\boldsymbol{x})=\|\varphi(\boldsymbol{x})\|\frac{\varphi(\boldsymbol{x})}{\|\varphi(\boldsymbol{x})\|}=\|\varphi(\boldsymbol{x})\|\frac{grad\ ln\ p(\boldsymbol{x})}{\|grad\ ln\ p(\boldsymbol{x})\|}=\|\varphi(\boldsymbol{x})\|(\tilde{x}_1/\|\tilde{\boldsymbol{x}}\|,\tilde{x}_2/\|\tilde{\boldsymbol{x}}\|) \tag{55}$$

where $\tilde{x}_1\equiv(\partial\varepsilon(\boldsymbol{x})/\partial x_1)\|\boldsymbol{x}\|+\varepsilon(\boldsymbol{x})x_1/\|\boldsymbol{x}\|$, $\tilde{x}_2\equiv(\partial\varepsilon(\boldsymbol{x})/\partial x_2)\|\boldsymbol{x}\|+\varepsilon(\boldsymbol{x})x_2/\|\boldsymbol{x}\|$ and $\tilde{\boldsymbol{x}}\equiv(\tilde{x}_1,\tilde{x}_2)$

Taking divergence of the both hand sides in equation (55) yields;

$$\begin{aligned}div\,\varphi(\boldsymbol{x})&=grad\,\|\varphi(\boldsymbol{x})\|(\tilde{x}_1/\|\tilde{\boldsymbol{x}}\|,\tilde{x}_2/\|\tilde{\boldsymbol{x}}\|)+\|\varphi(\boldsymbol{x})\|\,div\,(\tilde{x}_1/\|\tilde{\boldsymbol{x}}\|,\tilde{x}_2/\|\tilde{\boldsymbol{x}}\|)\\&=q(\boldsymbol{x})b(\boldsymbol{x})-c(\boldsymbol{x})h(\boldsymbol{x})-\varepsilon(\boldsymbol{x})\|\varphi(\boldsymbol{x})\|\end{aligned} \tag{56}$$

Hence the following equation is obtained.

$$grad\,\|\varphi(\boldsymbol{x})\|(\tilde{x}_1/\|\tilde{\boldsymbol{x}}\|,\tilde{x}_2/\|\tilde{\boldsymbol{x}}\|)=q(\boldsymbol{x})b(\boldsymbol{x})-c(\boldsymbol{x})h(\boldsymbol{x})-\|\varphi(\boldsymbol{x})\|\,div\tilde{\boldsymbol{x}}/\|\tilde{\boldsymbol{x}}\| \tag{57}$$

$$\therefore\ \frac{\tilde{x}_1}{\|\tilde{\boldsymbol{x}}\|}\frac{\partial\|\varphi(\boldsymbol{x})\|}{\partial x_1}+\frac{\tilde{x}_2}{\|\tilde{\boldsymbol{x}}\|}\frac{\partial\|\varphi(\boldsymbol{x})\|}{\partial x_2}=q(\boldsymbol{x})b(\boldsymbol{x})-c(\boldsymbol{x})h(\boldsymbol{x})-\|\varphi(\boldsymbol{x})\|(\varepsilon(\boldsymbol{x})+div\,\tilde{\boldsymbol{x}}/\|\tilde{\boldsymbol{x}}\|) \tag{58}$$

The left hand side in equation (58) stands for a directional derivative with respect to the volume of commodity transportation. Its direction is obtained by normalizing $(\tilde{x}_1,\tilde{x}_2)$, and a change in commodity flow volume in the direction where the transport cost gets higher is larger than in other directions.

The first term in the right hand side expresses the supply of commodities at location $\boldsymbol{x}$, the second term depicts household consumption and finally the third term implies the depreciation in goods in transporting and *generalized diffusion effect*. Here generalized diffusion effect means that a change in shipment of commodities gets smaller as the location of commodities transported gets further from the city center. This fact never appears unless SCVF and land density are taken into account.

Although $div\,\tilde{\boldsymbol{x}}/\|\tilde{\boldsymbol{x}}\|$ can be expanded, it becomes a very complex formula and it is not easy to see its economic implication. However when a change in $\varepsilon(\boldsymbol{x})$ is small as compared with the distance between $\boldsymbol{x}$ and the city center, $div\,\tilde{\boldsymbol{x}}/\|\tilde{\boldsymbol{x}}\|$ becomes a decreasing function of distance showing a property of diffusion effect.

The right hand side in equation (58) is a scalar. Therefore if $\tilde{x}_1/\|\tilde{\boldsymbol{x}}\|$ in the left hand side is large (i.e. the transport cost is high in the direction of x_1), it is natural that $\partial\|\varphi(\boldsymbol{x})\|/\partial x_1$ gets smaller. That is, the volume of commodities transported becomes smaller in the direction where the transport cost gets higher. This matches our sense in observing usual transportation phenomena.

Then the system of characteristic differential equations for equation (58) is as follows: (note that parameter s appears in this and next sections, but its meaning is different from one in the previous section.)

$$\frac{dx_1(s)}{ds} = \tilde{x}_1(s)/\|\tilde{\boldsymbol{x}}(s)\| \tag{59}$$

$$\frac{dx_2(s)}{ds} = \tilde{x}_2(s)/\|\tilde{\boldsymbol{x}}(s)\| \tag{60}$$

$$\frac{d\|(\varphi_1(x_1(s),x_2(s)),\varphi_2(x_1(s),x_2(s))\|}{ds} = q(\boldsymbol{x})b(\boldsymbol{x}) - c(\boldsymbol{x})h(\boldsymbol{x}) - \|\varphi(\boldsymbol{x})\|(\varepsilon(\boldsymbol{x}) + div\,\tilde{\boldsymbol{x}}/\|\tilde{\boldsymbol{x}}\|) \tag{61}$$

Taking into account that the direction of $\varphi(\boldsymbol{x})$ is the same as that of *grad ln p*($\boldsymbol{x}$), we set initial conditions for equations (59) to (61) as $x_1(0)=0, x_2(0)=0,$ and $\|(\varphi_1(x_1(0),x_2(0),\varphi_2(x_1(0),x_2(0))\|) = \|\varphi_0\|$, since we assume a simple von Thünen ring. From equations (59) and (60), the following differential equation is obtained.

$$\left(\frac{dx_1(s)}{ds}\right)^2 + \left(\frac{dx_2(s)}{ds}\right)^2 = 1 \tag{62}$$

Now solving equation (58) introducing a distance parameter s with the initial value of zero at the city center, we obtain equation (63).

$$\begin{aligned}\|\varphi(\boldsymbol{x}(s))\| &= \int_0^s [q(\boldsymbol{x}(\tau))b(\boldsymbol{x}(\tau)) - c(\boldsymbol{x}(\tau))h(\boldsymbol{x}(\tau))] \\ &\cdot exp\left(-\int_\tau^s div\,\tilde{\boldsymbol{x}}(\eta)/\|\tilde{\boldsymbol{x}}(\eta)\|\,d\eta\right) exp\left(-\int_\tau^s \varepsilon(\boldsymbol{x}(\eta))\,d\eta\right) d\tau\end{aligned} \tag{63}$$

Moreover the vector of commodity flow is denoted by;

$$\varphi(\boldsymbol{x}) = (\|\varphi(\boldsymbol{x})\|\tilde{x}_1/\|\tilde{\boldsymbol{x}}\|,\ \|\varphi(\boldsymbol{x})\|\tilde{x}_2/\|\tilde{\boldsymbol{x}}\|) \tag{64}$$

If SCVF is even, the direction of commodity shipment shows a line. But when SCVF is uneven, the direction will illustrate a curve.

Now let us explain the economic implication of equation (63). [·] in the integral is commodities produced minus household consumption. The second term describes the generalized diffusion effect in which the volume of commodities transported to the point $\boldsymbol{x}$ more decreases as $\boldsymbol{x}$ gets further from the city center. This is the first theoretical finding among literature, and it never appears without variable land density and SCVF.

The last term is an exponential integral of the variable transport cost showing that the commodities are depreciated by transport services. Incidentally if ε is a constant, then the last term becomes *exp* (-$\varepsilon\cdot(s-\tau)$) and one can intuitively see its economic implication. The equation (63) can be described in words as follows:

commodity volume transported to point $\boldsymbol{x}$ = (*total supply of commodities – total consumption of commodities*) ×
generalized diffusion effect – depreciation in commodities by transport services

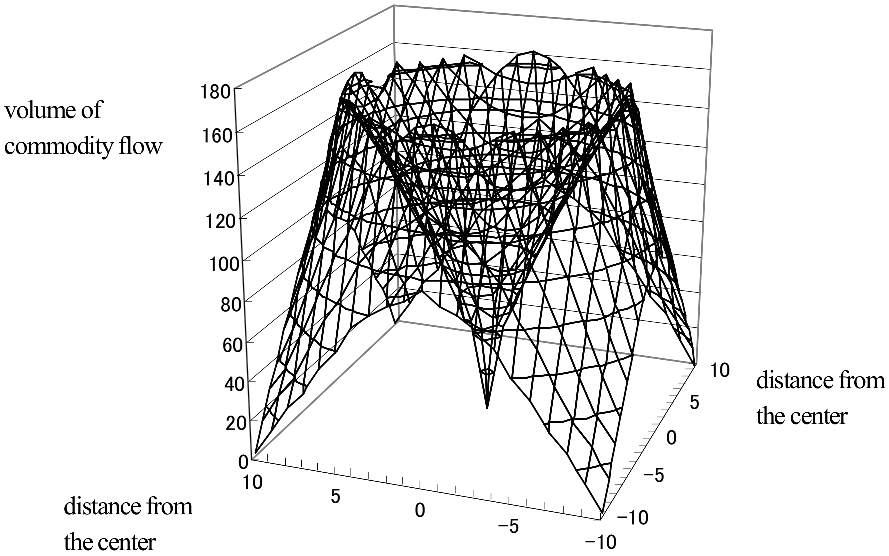

Figure 3 Shape of the Volume of Commodity Flow

15. THE OPTIMAL TRANSPORT OF LABOR

Regarding the optimal transport of labor, it can also be solved applying the same method in the case of commodity transport. Since $w(\boldsymbol{x})b\dfrac{\psi(\boldsymbol{x})}{\|\psi(\boldsymbol{x})\|}=grad\ w(\boldsymbol{x})$, we obtain $\dfrac{\psi(\boldsymbol{x})}{\|\psi(\boldsymbol{x})\|}=\dfrac{grad\ ln\ w(\boldsymbol{x})}{\|grad\ ln\ w(\boldsymbol{x})\|}=.$ $-(\hat{x}_1\|\hat{\boldsymbol{x}}\|,\hat{x}_2\|\hat{\boldsymbol{x}}\|)$.

where $\hat{x}_1\equiv(\partial\varsigma(\boldsymbol{x})/\partial x_1)\|\boldsymbol{x}\|+\varsigma(\boldsymbol{x})x_1/\|\boldsymbol{x}\|$, $\hat{x}_2\equiv(\partial\varsigma(\boldsymbol{x})/\partial x_2)\|\boldsymbol{x}\|+\varsigma(\boldsymbol{x})x_2/\|\boldsymbol{x}\|$ and $\hat{\boldsymbol{x}}\equiv(\hat{x}_1,\hat{x}_2)$

$$\therefore\psi(\boldsymbol{x})=\|\psi(\boldsymbol{x})\|\frac{\psi(\boldsymbol{x})}{\|\psi(\boldsymbol{x})\|}=-\|\psi(\boldsymbol{x})\|(\hat{x}_1/\|\hat{\boldsymbol{x}}\|,\hat{x}_2/\|\hat{\boldsymbol{x}}\|)\tag{65}$$

Taking divergence of the both hand sides in equation (65), we have;

$$div\,\psi(\boldsymbol{x})=grad\|\psi(\boldsymbol{x})\|(\hat{x}_1/\|\hat{\boldsymbol{x}}\|,\hat{x}_2/\|\hat{\boldsymbol{x}}\|)+\|\psi(\boldsymbol{x})\|\,div\,(\hat{x}_1/\|\hat{\boldsymbol{x}}\|,\hat{x}_2/\|\hat{\boldsymbol{x}}\|)\tag{66}$$

$$\begin{aligned}\therefore grad\|\psi(\boldsymbol{x})\|(\hat{x}_1/\|\hat{\boldsymbol{x}}\|,\hat{x}_2/\|\hat{\boldsymbol{x}}\|)&=div\,\psi(\boldsymbol{x})-\|\psi(\boldsymbol{x})\|\,div\,(\hat{x}_1/\|\hat{\boldsymbol{x}}\|,\hat{x}_2/\|\hat{\boldsymbol{x}}\|)\\&=l_sh(\boldsymbol{x})-l_d(\boldsymbol{x})b(\boldsymbol{x})-\varsigma(\boldsymbol{x})\|\psi(\boldsymbol{x})\|+\|\psi(\boldsymbol{x})\|\,div\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|\end{aligned}\tag{67}$$

$$\therefore\frac{\hat{x}_1}{\|\hat{\boldsymbol{x}}\|}\frac{\partial\|\psi(\boldsymbol{x})\|}{\partial x_1}+\frac{\hat{x}_2}{\|\hat{\boldsymbol{x}}\|}\frac{\partial\|\psi(\boldsymbol{x})\|}{\partial x_2}=l_sh(\boldsymbol{x})-l_d(\boldsymbol{x})b(\boldsymbol{x})-\|\psi(\boldsymbol{x})\|(\varsigma(\boldsymbol{x})-div\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|)\tag{68}$$

The interpretation of equation (68) is as follows: The left hand side is a directional derivative with respect to the volume of labor commuting. Its direction is the normalization of $(\hat{x}_1,\hat{x}_2)$, and a change in the volume of labor commuting flow in the direction where the transport cost gets higher is larger than in other directions. While the first term in the right hand side is a household labor supply at location $\boldsymbol{x}$, the second is firms' labor demand, $\zeta(\boldsymbol{x})\cdot\ \|\psi(\boldsymbol{x})\|$ is the transport cost in conveying labor, and finally $\|\psi(\boldsymbol{x})\|\,div\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|$ is *the generalized congestion effect*. That is, the change in labor flow becomes larger as the location of commuting gets closer to the center. This fact also never appears if SCVF and land density are not taken into account.

$div\,\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|$ can be expanded, however, it becomes a very complex formula and it is not easy to see its economic implication as well. When the distance between $\boldsymbol{x}$ and the city center is short, $div\,\hat{\boldsymbol{x}}/\|\hat{\boldsymbol{x}}\|$ takes a large value illustrating a characteristic of congestion effect.

The right hand side in equation (68) is a scalar. Therefore if $\hat{x}_1/\|\hat{x}\|$ in the left hand side is large (i.e. the transport cost is high in the direction of x_1), it is natural that $\partial\|\psi(\boldsymbol{x})\|/\partial x_1$ gets smaller. That is, the volume of labor transported becomes smaller in the direction where the transport cost gets higher. This also matches our sense in usual transport phenomena.

Then the system of characteristic differential equations for equation (68) is as follows:

$$\frac{dx_1(s)}{ds} = \hat{x}_1(s) / \| \hat{\boldsymbol{x}}(s) \| \tag{69}$$

$$\frac{dx_2(s)}{ds} = \hat{x}_2(s) / \| \hat{\boldsymbol{x}}(s) \| \tag{70}$$

$$\frac{d \| \psi(x_1(s), x_2(s)) \|}{ds} = l_s h(\boldsymbol{x}) - l_d(\boldsymbol{x}) b(\boldsymbol{x}) - \| \psi(x_1(s), x_2(s)) \| (\varsigma(\boldsymbol{x}) - div\hat{\boldsymbol{x}} / \| \hat{\boldsymbol{x}} \|) \tag{71}$$

Taking into account that the direction of $\psi(\boldsymbol{x})$ is the same as that of *grad ln w*($\boldsymbol{x}$), we set the initial value as, $x_1(0) = 0, x_2(0) = 0, \text{and} \| (\psi_1(x_1(0), x_2(0)), \psi_2(x_1(0), x_2(0))) \| = \| \psi_0 \|$. From the equations (69) and (70), we obtain the following differential equation.

$$\left(\frac{dx_1(s)}{ds} \right)^2 + \left(\frac{dx_2(s)}{ds} \right)^2 = 1 \tag{72}$$

Equation (72) can be solved as $x_1(s) = \gamma_1(S\text{-}s)$ and $x_2(s) = \gamma_2(S\text{-}s)$ ($\gamma_1^2 + \gamma_2^2 = 1$ and S is the parameter value which sets $x_1(S)$ and $x_2(S)$ on the city boundary.) referring to the initial conditions. Similar to the case of commodities, we get the solution in equation (68) as follows:

$$\begin{aligned} \| \psi(\boldsymbol{x}(S-s)) \| = & \int_0^s [l_s h(\boldsymbol{x}(S-\tau)) - l_d(\boldsymbol{x}(S-\tau)) b(\boldsymbol{x}(S-\tau))] \\ & exp\left(\int_\tau^s div\, \hat{\boldsymbol{x}}(S-\eta) / \| \hat{\boldsymbol{x}}(S-\eta) \| \, d\eta \right) exp\left(-\int_\tau^s \varsigma(\boldsymbol{x}(S-\eta)) \, d\eta \right) d\tau \end{aligned} \tag{73}$$

The vector of labor commuting flow is expressed as follows:

$$\psi(\boldsymbol{x}) = (-\| \psi(\boldsymbol{x}) \| \hat{x}_1 / \| \hat{\boldsymbol{x}} \|, \; -\| \psi(\boldsymbol{x}) \| \hat{x}_2 / \| \hat{\boldsymbol{x}} \|) \tag{74}$$

If SCVF is even, the direction of labor commuting shows a line. But when SCVF is uneven, the direction will depict a curve like in the commodity flow.

Let us explain the economic implication of equation (73). [·] in the integral is household labor supply minus labor demand by firms. The second term describes the generalized congestion effect where the volume of workers commuting to the point $\boldsymbol{x}$ more increases as $\boldsymbol{x}$ gets closer to the city center. This is also the first theoretical finding among literature, and it never appears without variable land density and SCVF.

The last term is an exponential integral of the variable transport cost showing that the labor forces are reduced by transport services. By the way if ζ is a constant, then the last term becomes *exp* ($-\zeta \cdot (s-\tau)$) and one can intuitively see the transportation cost. The equation (73) can also be written in words as follows:

the number of households commuting to point $\boldsymbol{x}$ = (*total supply of labor – total demand for labor*)× *generalized congestion effect – reduction in labor forces by transport services*

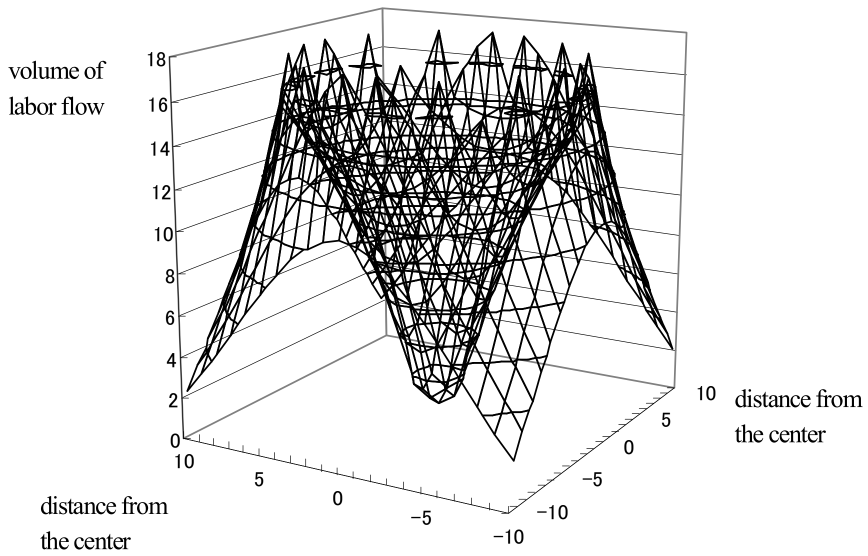

Figure 4 Shape of the Volume of Labor Flow

16. EQUILIBRIUM CONDITIONS

Equilibrium conditions in the model are summarized as follows:

Commodity market

$$\| \varphi(\boldsymbol{x}(S)) \| = 0 \tag{75}$$

Labor market

$$\| \psi(\boldsymbol{x}(0)) \| = 0 \quad \text{(the number of households commuting into the city center = 0)} \tag{76}$$

Capital market

$$ks(\boldsymbol{x}) = k_d(\boldsymbol{x})b(\boldsymbol{x}) \text{ and } \iint_A ks(\boldsymbol{x})dx_1dx_2 = N \cdot k_s \tag{77}$$

Land market: equations (15) to (18)

Location conditions

$$\pi_B(\boldsymbol{x}) = 0 : \boldsymbol{x} \in \text{business district} \tag{78}$$

$$u(\boldsymbol{x}) = u^* : \boldsymbol{x} \in \text{residential area} \tag{79}$$

The numbers of firms and households:

$$M = \iint_{A_B} \frac{1}{m_B(\boldsymbol{x})} dx_1dx_2 \quad (A_B : \text{business district}) \tag{80}$$

$$N = \iint_{A_H} \frac{1}{m_H(\boldsymbol{x})} dx_1dx_2 \quad (A_H : \text{residential area}) \tag{81}$$

From these conditions, bid rents $g_B(\boldsymbol{x})$ and $g_H(\boldsymbol{x})$, bid max lot sizes $m_B(\boldsymbol{x})$ and $m_H(\boldsymbol{x})$ and the equilibrium utility level are obtained determining the equilibrium urban configuration.

17. INTRODUCING SPATIAL CHARACTERISTIC VECTOR FIELD AND EXPECTED RESULTS

As an example of SCVF, we consider a case where the transport costs increase at a certain place in the city. Figure 5 shows a change in heights from the city center to a point (7, 7). The land shape shows a slope from a circle passing a point (3. 3), and the height hits the peak of 4.5 at a point (7, 7). This change is expressed by a part of the normal distribution surface, and then the spatial configuration of commodity price shows a surface presented in Figure 6 hitting its peak at point (5,5).

In the slope area, the isoprice set is calculated as a circle of $(x_1\text{-}5)^2+(x_2\text{-}5)^2 = r^2$ differing from the isoprice set from the city center $x_1^2 + x_2^2 = r^2$. Thus transport routes become a set of curves from points $(3, x_2)$. Then commodity supply, household consumption, land use pattern etc. are determined under the general equilibrium framework.

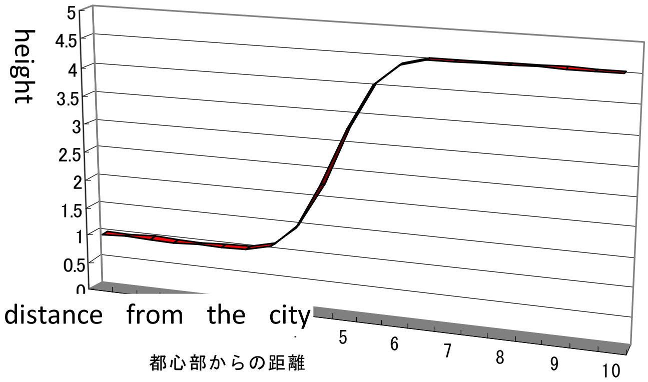

Figure 5. Hypothetical Geographical Condition

Moreover Figure 7 illustrates an expected result by our approach. Due to a heterogeneous geographical condition, unit transport cost varies depending on different points. So commodity prices do not change uniformly resulting in curved transport routes. This expectation suggests internalization of a transport network which more approximates the real transport networks than in the traditional urban economic models.

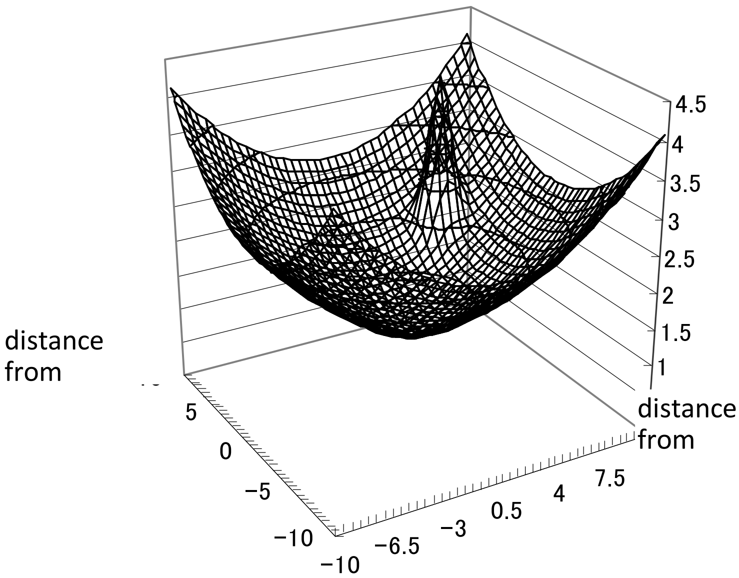

Figure 6. A Change in Spatial Configuration of Commodity Price

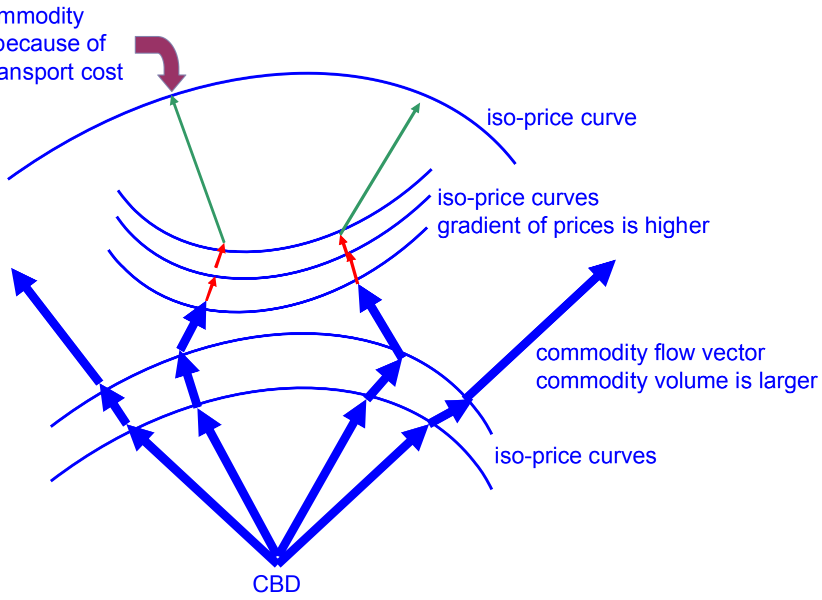

Figure 7. Expected Transport Routes and Commodity Volume

18. CONCLUDING REMARKS

This paper has examined the patterns of commodity and labor transportation in a monocentric city by solving the partial differential equations under the framework initiated by Beckmann (1952) and Beckmann and Puu (1985). This study extends my previous achievement into a case of variable von Thünen transport coefficient introducing SCVF. The new findings in this paper are summarized as follows:

(1)Transport networks can be internalized by applying the variational principle.

(2)Although only linear routes in a network appear over an even SCVF, curved routes in a network can be internalized over an uneven SCVF.

(3)By introducing uneven SCVF, generalized diffusion effect and generalized congestion effect can be obtained.

(4)Commodity and labor flows irregularly change over an uneven SCVF.

(5)A curved route relates the light changing in a continuously uneven medium studied in physics. Relationship our approach and the wave equation could be an interesting theme.

Areas worth examining in the near future include solving the full equilibrium model mentioned in this study with appropriate hypothetical parameter value. Empirical studies based on our theory are an interesting topic as well.

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